数理解析研究所講究録
1123 巻 2000 年 76-82

Palais-Smale Condition for Some Semilinear Parabolic Equations

池畠 良
Ryo IKEHATA
Department of Mathematics, Faculty of School Education
Hiroshima University, Higashi-Hiroshima 739-8524, Japan

1 Introduction

In this paper we are concerned with the following mixed problem to semilinear parabolic equation:

$$u_t(t,x) - \Delta u(t,x) = |u(t,x)|^{p-1}u(t,x), \ \ (t,x) \in (0,T) \times \Omega, \tag{1}$$

$$u(0,x) = u_0(x), \ \ x \in \Omega, \tag{2}$$

$$u|_{\partial\Omega} = 0, \ \ t \in (0,T). \tag{3}$$

Here, $1 < p \leq \frac{N+2}{N-2}$, $\Omega \subset R^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$. In the case when $1 < p < \frac{N+2}{N-2}$, of course, we can treat the low dimensional case $N = 1, 2$, but for simplicity we restrict our attention to the above mentioned case. For large initial data u_0 in some sense, it is well-known that the solution $u(t,x)$ to the problem (1)-(3) blows up in a finite time (see Ikehata-Suzuki[7], Ishii[9], Levine[10], Ôtani[11], Tsutsumi[16], and Payne-Sattinger[12]), meanwhile for small initial data, exponentially decaying solutions are obtained (see [7] and the references therein). In this paper, we have much interest in solutions to (1)-(3) which neither blowup nor decay. In that occasion, we proceed our argument based on the following local well-posedness theorem due to [7] (see also, Hoshino-Yamada[5]). In the following, $\|\cdot\|_q (1 \leq q \leq \infty)$ means the usual (real) $L^q(\Omega)$-norm.

Proposition 1. 1 *For each $u_0 \in H_0^1(\Omega)$, there exists a number $T_m > 0$ such that the problem (1.1)-(1.3) has a unique solution $u \in C([0,T_m); H_0^1(\Omega))$ which becomes classical on $(0,T_m)$. Furthermore, if $T_m < +\infty$, then*

$$\lim_{t \uparrow T_m} \|u(t,\cdot)\|_\infty = +\infty,$$

and in particular, in the case when $1 < p < \frac{N+2}{N-2}$ one also has

$$\lim_{t \uparrow T_m} \|\nabla u(t,\cdot)\|_2 = +\infty.$$

Set

$$X = H_0^1(\Omega),$$

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

$$I(u) = \|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1},$$
$$\mathcal{N} = \{v \in X \setminus \{0\} | I(v) = 0\},$$
$$d_p = \inf_{v \in \mathcal{N}} J(v) = \inf\{\sup_{\lambda \geq 0} J(\lambda v)| \ v \in X \setminus \{0\}\}.$$

It is easy to show that the potential depth d_p (see Sattinger[13]) satisfies $d_p > 0$ because of the Sobolev continuous embedding $X \hookrightarrow L^{p+1}(\Omega)$ $(1 < p \leq \frac{N+2}{N-2})$. The stable and unstable sets are defined as usual:

$$W = \{u \in X | J(u) < d_p, I(u) > 0\} \cup \{0\},$$

$$V = \{u \in X | J(u) < d_p, I(u) < 0\}.$$

Furthermore, for later use we define the following notations.

$$E = \{u \in X| - \Delta u = |u|^{p-1}u \ in \ \Omega, u|_{\partial\Omega} = 0\},$$

$$E^* = \{u \in D^{1,2}(R^N)| - \Delta u = |u|^{p-1}u \ in \ R^N\},$$

$$E_+^* = \{u \in E^*| \ u \geq 0 \ in \ R^N\},$$

$$J_*(u) = \frac{1}{2}\int_{R^N} |\nabla u(x)|^2 dx - \frac{1}{p+1}\int_{R^N} |u(x)|^{p+1} dx.$$

Here $D^{1,2}(R^N)$ denotes the closure of $C_0^\infty(R^N)$ with respect to the norm $\|\nabla u\|_{L^2(R^N)}$. In particular, in the case when $p = \frac{N+2}{N-2}$, because of the Sobolev embedding $S\|u\|_{L^{p+1}(R^N)} \leq \|\nabla u\|_{L^2(R^N)}$ for $u \in D^{1,2}(R^N)$, one also has

$$d^* = \inf\{\sup_{\lambda \geq 0} J_*(\lambda v)| \ v \in D^{1,2}(R^N) \setminus \{0\}\} = \frac{1}{N}S^N > 0.$$

Note that $d^* = d_p$ with $p = \frac{N+2}{N-2}$.

Remark 1. 1 *In the case when $p = \frac{N+2}{N-2}$, it is well-known (Struwe[14]) that the family $\{u_\varepsilon^*(x)\}$ such as*

$$u_\varepsilon^*(x) = \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{N-2}{2}}}, \quad \varepsilon > 0$$

satisfies

$$-\Delta u = |u|^{p-1}u \ in \ R^N,$$

so that $E_+^* \setminus \{0\} \neq \emptyset$.

By the way, quite recently, in [7] the following result has been shown with regard to the singularity of a global solution to the problem (1)-(3) under the assumptions below: let $u(t,x)$ be a solution to (1.1)-(1.3) as in Proposition 1.1. Furthermore, one assumes that

(A.1) $u_0 \geq 0$.

(A.2) $p = \frac{N+2}{N-2}$.

(A.3) $\Omega = \{x \in R^N| \ |x| < 1\}$.

(A.4) $u(t,x)=u(t,|x|)$, $u_r(t,r)<0$ on $0<r\leq 1$ with $r=|x|$.

Finally, assume $T_m=+\infty$. For $1<p\leq \frac{N+2}{N-2}$ set

$$C_0=\frac{2(p+1)}{p-1}\lim_{t\to+\infty} J(u(t,\cdot)). \tag{4}$$

Note that $C_0\geq 0$ if $T_m=+\infty$ (see [10]). Then, their results read as follows.

Theorem 1. 1 *([7]) Assume (A.1)-(A.4). Let $u(t,x)$ be a solution to (1)-(3) on $[0,T_m)$ as in Proposition 1.1. Suppose $T_m=+\infty$ and $C_0>0$. Then, there exists a sequence $\{t_n\}$ with $t_n\to+\infty$ as $n\to+\infty$ such that*

(1) $|\nabla u(t_n,x)|^2\to C_0\delta_0$ (weakly-) in $C_0(\Omega)^*$,*

(2) $u(t_n,x)^{p+1}\to C_0\delta_0$ (weakly-) in $C_0(\Omega)^*$,*

as $n\to+\infty$. Here, δ_0 means the usual Dirac measure having a unit mass at the origin.

Since $C_0>0$ if and only if $u(t,\cdot)\notin(W\cup V)$ for all $t\geq 0$, their theorem states that a global orbit $u(t,\cdot)$ which neither decay nor blowup (if this kind of solution can be constructed!) have a strong singularity at the origin. In connection with this result, we have just noticed that for such a sequence $\{t_n\}$ constructed in Theorem 1.1 above, $\{u(t_n,\cdot)\}$ becomes a Palais-Smale sequence so that the global compactness result due to Struwe[15] can be applied to this functional sequence. Our first result reads as follows:

Theorem 1. 2 *Let $\{u(t_n,\cdot)\}$ be a sequence as in Theorem 1.1. Under the same assumptions as in Theorem 1.1, there exist an integer $k\in N$, a sequence of radii $\{R_n^i\}$ with $\lim_{n\to+\infty}R_n^i=+\infty$, a sequence $\{x_n^i\}\in\Omega$, and $u^i\in E_+^*\setminus\{0\}(1\leq i\leq k)$ such that (taking a subsequence)*

$$\lim_{n\to+\infty}\|\nabla(u(t_n,\cdot)-\sum_{i=1}^k u_n^i)\|_{L^2(R^N)}=0,$$

$$\lim_{t\to+\infty}J(u(t,\cdot))=\lim_{n\to+\infty}J(u(t_n,\cdot))=kd^*,$$

$$\lim_{n\to+\infty}\|\nabla u(t_n,\cdot)\|_2^2=\sum_{i=1}^k\|\nabla u^i\|_{L^2(R^N)}^2=kS^N,$$

where

$$u_n^i(x)=(R_n^i)^{\frac{N-2}{2}}u^i(R_n^i(x-x_n^i))\ (1\leq i\leq k),\ \ n=1,2,\cdots.$$

Remark 1. 2 *By considering scaling and translation, one finds that the compactness of $\{u(t_n,\cdot)\}$ destroyed in Theorem 1.1 is restored once more. On the other hand, for the proof of this Theorem, we have to notice the following fact (see [14]) that each u^i is of the form $u^i(x)=u_\varepsilon^*(x)$ (see Remark 1.1) with some ε and satisfies $J_*(u^i)=d^*$ (least energy level).*

Remark 1. 3 *Under the assumptions Ω = star-shaped and $u_0(x)\geq 0$, one can get the quite same results as in the radial case above. In the case when u_0 changes sign, however, even if Ω is star-shaped, one needs a few modifications of the results above (see [14]).*

The next result is concerned with the case when $1 < p < \frac{N+2}{N-2}$. It seems not to be known that any global solutions to (1)-(3) naturally contain a subsequence which is relatively compact in X in the subcritical case. Our second result reads as follows:

Theorem 1. 3 *Let $1 < p < \frac{N+2}{N-2}$ and $u(t,x)$ be a solution on $[0, T_m)$ as in Proposition 1.1. If $T_m = +\infty$, then there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ such that $\{u(t_n, \cdot)\}$ becomes relatively compact in X so that there exists an element $u_\infty \in E$ such that $u(t_n, \cdot) \to u_\infty$ in X as $n \to +\infty$ along a subsequence.*

Remark 1. 4 *In Theorem 1.3, if, in particular, $C_0 > 0$, then one has $u_\infty \in E \setminus \{0\}$. Furthermore, the construction of such a sequence $\{t_n\}$ is in the same way as in Theorem 1.2.*

2 Palais-Smale sequence

In this section, reviewing some results concerning Theorem 1.1 due to [7] we shall construct some Palais-Smale sequences of a global solution to the problem (1)-(3).

First, suppose $1 < p \le \frac{N+2}{N-2}$ and $T_m = +\infty$ in Proposition 1.1. Since its solution satisfies the energy identity:

$$J(u(t,\cdot)) + \int_0^t \|u_t(s,\cdot)\|_2^2 ds = J(u_0) \tag{5}$$

for all $t \ge 0$, this implies that the function $t \mapsto J(u(t,\cdot))$ is monotone decreasing so that $C_0 \ge 0$ (see (4)) is meaningfull. Letting $t \to +\infty$ in (5), the improper integral $\displaystyle\int_0^\infty \|u_t(s,\cdot)\|_2^2 ds$ is finite determined. Therefore, there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ such that

$$\lim_{n\to+\infty} \|u_t(t_n,\cdot)\|_2^2 = 0.$$

Note that this sequence $\{t_n\}$ coincides with the one in Theorem 1.1.

Next, multiplying the both sides of (1) by $u(t,x)$ and integrating it over Ω, we have

$$(u_t(t,\cdot), u(t,\cdot)) = -I(u(t,\cdot)), \tag{6}$$

where $(f,g) = \displaystyle\int_\Omega f(x)g(x)dx$. Because of [2], it holds true that $\|u(t,\cdot)\|_2 \le C$ for all $t \ge 0$ with some constant $C > 0$. Therefore, one has

$$|I(u(t_n,\cdot))| \le C\|u_t(t_n,\cdot)\|_2$$

for all $n \in N$. Letting $n \to +\infty$, it follows that

$$\lim_{n\to+\infty} I(u(t_n,\cdot)) = 0. \tag{7}$$

On the other hand, the identity holds good:

$$J(u) = \frac{p-1}{2(p+1)}\|\nabla u\|_2^2 + \frac{1}{p+1}I(u). \tag{8}$$

So, from (8) with $u = u(t_n,\cdot)$ and (6)-(7) we find that

Lemma 2. 1 *Let $u(t,\cdot)$ be as in Proposition 1.1. If $T_m = +\infty$, then there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to \infty$ such that*

$$\lim_{n\to+\infty} \|u_t(t_n,\cdot)\|_2 = 0,$$

$$\lim_{n\to+\infty} \|\nabla u(t_n,\cdot)\|_2^2 = C_0,$$

$$\lim_{n\to+\infty} \|u(t_n,\cdot)\|_{p+1}^{p+1} = C_0.$$

From this lemma, one obtains the next ones:

Lemma 2. 2 *Let $u(t,x)$ be a local solution constructed in Proposition 1.1. If $T_m = +\infty$, then there exists a Palais-Smale sequence to the problem (1)-(3).*

Proof. Let $\{t_n\}$ be as in Lemma 2.1. Then, it follows that

$$J(u_0) \geq J(u(t_n,\cdot)) \to \frac{p-1}{2(p+1)} C_0 \geq 0 \ \ as \ \ n \to +\infty. \tag{9}$$

Furthermore, for such sequence, since $J \in C^1(X,R)$, by equation (1) we have

$$J'(u(t_n,\cdot))[v] = -(u_t(t_n,\cdot),v)$$

for each $v \in X$, where $J'(u) \in X^*$ means the usual Fréchet-derivative of J at $u \in X$. By this equality and the Schwarz inequality together with the Poincaré inequality one gets:

$$|J'(u(t_n,\cdot))[v]| \leq C_1 \|u_t(t_n,\cdot)\|_2 \|\nabla v\|_2$$

which implies

$$\|J'(u(t_n,\cdot))\|_{H^{-1}(\Omega)} \to 0(n \to +\infty), \tag{10}$$

where $C_1 > 0$ is a Poincaré constant. We find that $\{u(t_n,\cdot)\}$ becomes a Palais-Smale sequence because of (9) and (10). ∎

In particular, in the case when $p \in (1, \frac{N+2}{N-2})$ one gets the following compactness result. For the detailed proof, see the forthcoming paper [8].

Lemma 2. 3 *Suppose $p \in (1, \frac{N+2}{N-2})$. Let $u(t,x)$ be a global (i.e.,$T_m = +\infty$) solution to (1)-(3) as in Proposition 1.1. Then, the sequence $\{u(t_n,\cdot)\}$ constructed in Lemma 2.1 becomes relatively compact in X.*

Now, we are in a position to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. This result is a direct consequence of [14] (Theorem 3.1, p.184) and Lemma 2.2 and so, we shall omitt the details. But, since $\Omega = ball$, note that $E = \{0\}$ holds true in the present case. ∎

Proof of Theorem 1.3. The first half is a direct consequence of Lemma 2.3. In order to prove $u_\infty \in E$, note that the following estimates are proven:

$$\|f(u) - f(v)\|_{1+\frac{1}{p}} \leq p(\|u\|_{p+1} + \|v\|_{p+1})^{p-1} \|u-v\|_{p+1}$$

for all $u, v \in L^{p+1}(\Omega)$, and

$$|(f(u(t_n,\cdot)) - f(u_\infty), \phi)| \leq \|f(u(t_n,\cdot)) - f(u_\infty)\|_{1+\frac{1}{p}} \|\phi\|_{p+1}$$

for each $\phi \in C_0^\infty(\Omega)$, where $\{u(t_n,\cdot)\}$ is a sequence constructed in the first half. By combining these estimates with Lemma 2.1 and the Sobolev embedding $X \hookrightarrow L^{p+1}(\Omega)$, one obtains the desired statement. ∎

From Lemma 2.1 one has a result reviewed from the view point of the Palais-Smale condition.

Corollary 2. 1 *Let $1 < p \leq \frac{N+2}{N-2}$ and $u(t,x)$ be a global solution constructed in Proposition 1.1, i.e., $T_m = +\infty$. If $C_0 = 0$, then the sequence $\{u(t_n,\cdot)\}$ stated in Lemma 2.1 becomes relatively compact, and in fact, $u(t,\cdot) \to 0$ in X as $t \to +\infty$.*

From Theorem 1.1 and Corollary 2.1 with $p = \frac{N+2}{N-2}$, one can say that it depends on the least energy level $\dfrac{p-1}{2(p+1)}C_0$ whether the Palais-Smale condition holds good or not to the sequence $\{u(t_n,\cdot)\}$ as in Lemma 2.1.

Finally in this section, we shall apply Theorem 1.3 and Lemma 2.2 for the finite time blowup problem concerning (1)-(3). First, as a consequence of [14] one obtains the following lemma.

Lemma 2. 4 *Let Ω be a bounded smooth domain and $p = \frac{N+2}{N-2}$. Then, for all $v \in E$, one has $J(v) \in \{0\} \cup (d^*, +\infty)$, and also, for each $w \in E^* \setminus \{0\}$, one has $J_*(w) \in \{d^*\} \cup (2d^*, +\infty)$.*

The following result gives a kind of alternative proof of [11] concerning blowup problem.

Proposition 2. 1 *Let $1 < p \leq \frac{N+2}{N-2}$ and $u(t,x)$ be a local solution of (1)-(3) on $[0, T_m)$ constructed in Proposition 1.1. If $u(t_0,\cdot) \in V$ for some $t_0 \in [0, T_m)$, then $T_m < +\infty$.*

Proof. First, we shall deal with the case when $1 < p < \frac{N+2}{N-2}$. Suppose $T_m = +\infty$. Then, it follows from Theorem 1.3 that there exist a Palais-Smale sequence $\{u(t_n,\cdot)\}$ to the problem (1)-(3) and $u_\infty \in E$ such that $u(t_n,\cdot) \to u_\infty$ in X along a subsequence. On the other hand, it is well-known (see [6]) that $u(t,\cdot) \in V$ for all $t \in [t_0, \infty)$. Since W is a neighbourhood of 0 in X, if $u_\infty = 0$, then $u(t_m,\cdot) \in W$ holds with some t_m and this contradicts the fact that $W \cap V = \emptyset$. Thus, $u_\infty \in E \setminus \{0\}$. Because of the monotone decreasingness of a function $t \mapsto J(u(t,\cdot))$, one obtains $J(u(t_n,\cdot)) \geq J(u_\infty) \geq d_p$ which contradicts $u(t_n,\cdot) \in V$ with large t_n.

Next, we are concerned with the critical case $p = \frac{N+2}{N-2}$. Suppose $T_m = +\infty$. Obviously, $C_0 > 0$ holds true. Then, from Lemma 2.2 and Theorem 3.1 of [14], p.184 that there exist a Palais-Smale sequence $\{u(t_n,\cdot)\}$, $k \in N$, $u^0 \in E$, and $u^i \in E^* \setminus \{0\}$ $(1 \leq i \leq k)$ such that

$$\lim_{n\to+\infty} J(u(t_n,\cdot)) = \lim_{t\to+\infty} J(u(t,\cdot)) = J(u^0) + \sum_{i=1}^{k} J_*(u^i).$$

By Lemma 2.4 and the monotone decreasingness of a function $t \mapsto J(u(t,\cdot))$, one finds that

$$J(u(t,\cdot)) \geq d^*$$

for all $t \geq 0$. This contradicts also $u(t,\cdot) \in V$ for all $t \geq t_0$. ∎

References

[1] H. Brezis and L. Nirenberg. *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents.* Comm. Pure Appl. Math. **36**(1983), 437–477.

[2] T. Cazenave and P. L. Lions. *Solutions globales d'equations de la chaleur semi lineaires.* Comm. Partial Differential Equations **9**(1984), 955–978.

[3] Y. Giga. *A bound for global solutions of semilinear heat equations.* Comm. Math. Phys. **103**(1986),415–421.

[4] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Lecture Notes in Math. **840**, Springer-Verlag, 1981.

[5] H. Hoshino and Y. Yamada. *Solvability and smoothing effect for semilinear parabolic equations.* Funk. Ekva. **34**(1991), 475–494.

[6] R. Ikehata and T. Suzuki. *Stable and unstable sets for evolution equations of parabolic and hyperbolic type.* Hiroshima Math. J. **26**(1996), 475–491.

[7] R. Ikehata and T. Suzuki. *Semilinear parabolic equations involving critical Sobolev exponent: local and asymptotic behavior of solutions.* to appear in Diff. Int. Equations.

[8] R. Ikehata. *Remarks on the Palais-Smale condition for some semilinear parabolic equations.* preprint(1998).

[9] H. Ishii. *Asymptotic stability and blowing-up of solutions of some nonlinear equations.* J. Diff. Eq. **26**(1977), 291–319.

[10] H. A. Levine. *Some nonexistence and instability theorems of solutions of formally parabolic equations of the form $Pu_t = -Au+F(u)$.* Arch. Rat. Mech. Math. **51**(1973), 371–386.

[11] M. Ôtani. *Existence and asymptotic stability of strong solutions of nonlinear evolution equations with a difference term of subdifferentials.* Colloq. Math. Soc. Janos Bolyai, Qualitative Theory of Differential Equations **30**, North-Holland, Amsterdam, 1980.

[12] L. E. Payne and D. H. Sattinger. *Saddle points and unstability of nonlinear hyperbolic equations.* Israel J. Math. **22**(1975), 273–303.

[13] D. H. Sattinger. *On global solution of nonlinear hyperbolic equations.* Arch. Rat. Mech. Math. **30**(1968), 148–172.

[14] M. Struwe. *Variational Methods* A Series of Modern Surveys in Mathematics **34**, Springer-Verlag, 1996.

[15] M. Struwe. *A global compactness result for elliptic boundary value problems involving limiting nonlinearities.* Math. Z. **187**(1984), 511–517.

[16] M. Tsutsumi. *On solutions of semilinear differential equations in a Hilbert space.* Math. Japon. **17**(1972), 173–193.